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Short communication

Elliptical contact of thin biphasic cartilage layers: Exact solution for monotonic loading

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ABSTRACT

A three-dimensional unilateral contact problem for articular cartilage layers is considered in the framework of the biphasic cartilage model. The articular cartilages bonded to subchondral bones are modeled as biphasic materials consisting of a solid phase and a fluid phase. It is assumed that the subchondral bones are rigid and shaped like elliptic paraboloids. The obtained analytical solution is valid for monotonically increasing loading conditions.

0. Introduction

Biomechanical contact problems involving transmission of forces across biological joints are of considerable practical importance in surgery. Many solutions to the axisymmetric problem of contact interaction of articular cartilage surfaces in joints are available. Ateshian et al. (1994) obtained an asymptotic solution for the contact problem of two identical biphasic cartilage layers attached to two rigid impermeable spherical bones of equal radii modeled as elliptic paraboloids. Wu et al. (2000) extended this solution to a more general model by combining the assumption of the kinetic relationship from classical contact mechanics (Johnson, 1985) with the joint contact model for the contact of two biphasic cartilage (Ateshian et al., 1994). An improved solution for the contact of two biphasic cartilage layers which can be used for dynamic loading was obtained by Wu et al. (1997). These solutions have been widely used as theoretical background in modeling the articular contact mechanics. Recently, Mishuris and Argatov (2009) and Argatov and Mishuris (2010a) extended the analysis of Wu et al. (1996) by formulating the refined contact condition which takes into account the tangential displacements at the contact region.

When studying contact problems for real joint geometries, a numerical analysis, such as the finite element method, is necessary (Han et al., 2005), since exact analytical solutions were obtained only for two-dimensional (Ateshian and Wang, 1995), or axisymmetric and simple geometries (Eberhardt et al., 1990, 1991; Li et al., 1997). In this study, the axisymmetric model of articular contact mechanics developed by Ateshian et al. (1994) and Wu et al. (1996) is generalized for the three-dimensional case. The method developed by Argatov (2004) is used to obtain general relationships between the integral characteristics of the contact problem. The exact closed-form solution of the contact problem for biphasic cartilage layers attached to rigid bones shaped like elliptic paraboloids is obtained. Below we present only the essential points of our analytical constructions. For more details the reader is referred to the preprint (Argatov and Mishuris, 2010b).

1. Formulation of the contact problem

We consider a frictionless contact between two thin linear biphasic cartilage layers firmly attached to rigid bones shaped like elliptic paraboloids. Let $w_1(x_1,x_2,t)$ and $w_2(x_1,x_2,t)$ be the absolute values of the vertical displacements of the boundary points of the biphasic cartilage layers (see Fig. 1). Let us also denote by $\delta_0(t)$ the vertical approach of the bones. Then, the linearized contact condition in the contact region $\omega(t)$ can be written as

$$w_1(x_1,x_2,t)+w_2(x_1,x_2,t)=\delta_0(t)-\Phi(x_1,x_2), \tag{1}$$

where the function $\Phi(x_1,x_2)$ determines the initial gap between the cartilage surfaces. Using the usual approach of contact mechanics (Johnson, 1985), we can write

$$\Phi(x_1,x_2)=\frac{x_1^2}{2R_1}+\frac{x_2^2}{2R_2}, \tag{2}$$

where R_1 and R_2 are the principal relative radii of curvature defined by the equations

$$\frac{1}{R_1}+\frac{1}{R_2}=\frac{1}{R_1^{(1)}}+\frac{1}{R_2^{(1)}}+\frac{1}{R_1^{(2)}}+\frac{1}{R_2^{(2)}}, \tag{3}$$

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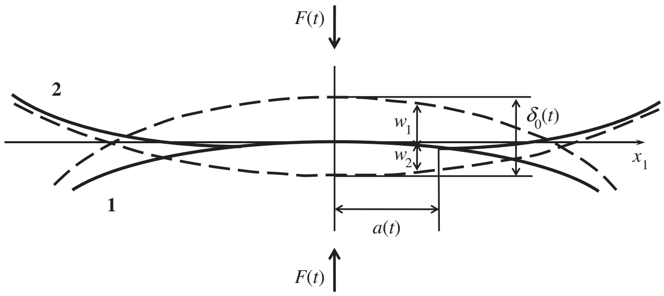

Fig. 1. Schematic diagram of the contact of articular cartilage surfaces 1 and 2 under load $F(t)$. The dashed lines imply the surfaces' profiles in the undeformed state; w_1 and w_2 are local deformations of the contacting surfaces 1 and 2, respectively; $\delta_0(t)$ is the relative approach between the subchondral bones covered with cartilages. The difference between the sum w_1+w_2 and $\delta_0(t)$ is due to the initial gap between the cartilage surfaces.

$$\frac{1}{R_2}-\frac{1}{R_1}=\left\{\left(\frac{1}{R_1^{(1)}}-\frac{1}{R_2^{(1)}}\right)^2+\left(\frac{1}{R_1^{(2)}}-\frac{1}{R_2^{(2)}}\right)^2 +2\left(\frac{1}{R_1^{(1)}}-\frac{1}{R_2^{(1)}}\right)\left(\frac{1}{R_1^{(2)}}-\frac{1}{R_2^{(2)}}\right)\cos 2\alpha\right\}^{1/2}. \qquad (4)$$

Here, $R_1^{(1)}, R_2^{(1)}$ and $R_1^{(2)}, R_2^{(2)}$ are the principal radii of curvature of the cartilage surfaces at the origin. The local coordinate systems are chosen in such a way that $R_1^{(1)} \geq R_2^{(1)}$ and $R_1^{(2)} \geq R_2^{(2)}$. Correspondingly, α is the angle between the axes of principal curvature, i.e., the local abscissa axes are inclined to each other by the angle α. Note that Eqs. (3) and (4) are valid also in the case of concave or saddle-shaped cartilage surfaces, if a negative sign is ascribed to the concave curvature.

Generalizing the asymptotic solution obtained by Ateshian et al. (1994) in the axisymmetric contact problem for biphasic layers, we will have

$$w_n(x_1,x_2,t)=\frac{h_n^3}{3\mu_{sn}}\left\{\Delta P(x_1,x_2,t)+\frac{3\mu_{sn}k_n}{h_n^2}\int_0^t \Delta P(x_1,x_2,\tau)\,d\tau\right\}. \qquad (5)$$

Here, μ_{sn} is the shear modulus of the solid phase of the cartilage tissue ($n=1,2$), h_1 and h_2 are the thicknesses of the cartilage layers, k_1 and k_2 are the cartilage permeabilities, $P(x_1,x_2,t)$ is the contact pressure, $\Delta=\partial^2/\partial x_1^2+\partial^2/\partial x_2^2$ is the Laplace operator. It should be noticed that the asymptotic formulas obtained by Ateshian et al. (1994) were given in the *dimensionless* form. Note also that as it was observed by Barry and Holmes (2001) and Ateshian et al. (1994) considered the small time scales, and Eq. (5) is not valid for long times compared with the typical diffusion time $\tau'_n=h_n^2/(k_nH_{An})$, where H_{An} is the aggregate modulus of the n-th cartilage layer given by $H_{An}=\lambda_{sn}+2\mu_{sn}$ with λ_s and μ_s being the Lamé coefficients of cartilage.

Substituting the expressions for the displacements $w_1(x_1,x_2,t)$ and $w_2(x_1,x_2,t)$ given by formula (5) into Eq. (1), we obtain the contact condition in the form

$$\Delta P(x_1,x_2,t)+\chi\int_0^t \Delta P(x_1,x_2,\tau)\,d\tau=m(\Phi(x_1,x_2)-\delta_0(t)), \qquad (6)$$

where $\chi=3(\mu_{s1}k_1/h_1^2+\mu_{s2}k_2/h_2^2)$ and $m=3(h_1^3/\mu_{s1}+h_2^3/\mu_{s2})^{-1}$.

Eq. (6) will be used to determine the contact pressure density $P(x_1,x_2,t)$. The contour $\Gamma(t)$ of the contact area $\omega(t)$ is determined from the condition that the contact pressure is positive and vanishes at the contour of the contact area. Moreover, because in a biphasic cartilage layer the contact pressure is carried primarily by the fluid phase, it is additionally assumed a smooth transition of the surface normal stresses from the contact region $(x_1,x_2)\in\omega(t)$ to the outside region $(x_1,x_2)\notin\omega(t)$ (Ateshian et al., 1994). Thus, we impose the following boundary conditions:

$$P(x_1,x_2,t)=0 \quad \text{and} \quad \frac{\partial P}{\partial n}(x_1,x_2,t)=0, \qquad (x_1,x_2)\in\Gamma(t). \qquad (7)$$

Here, $\partial/\partial n$ is the normal derivative. We will also assume that the density $P(x_1,x_2,t)$ is defined on the entire plane such that $P(x_1,x_2,t)=0$ for $(x_1,x_2)\notin\omega(t)$. Note that for determining the initial contact radius in the case of axisymmetric contact, Hlaváček (1999) suggested a new approach, which does not make use of the zero pressure gradient condition $(7)_2$ and is based on the analytical solutions of the axisymmetric contact problem for an incompressible single-phasic elastic layer (Matthewson, 1981).

The contact load $F(t)$ is related to the contact pressure by the equilibrium equation

$$\iint_{\omega(t)} P(x_1,x_2,t)\,dx_1\,dx_2=F(t). \qquad (8)$$

For non-decreasing loads when $dF(t)/dt\geq 0$, the contact zone increases. Thus, we assume that the following monotonicity condition holds: $\omega(t_1)\subset\omega(t_2)$ for $t_1\leq t_2$.

The aim of this study is to derive an exact solution of the contact problem for thin biphasic layers formulated by Eq. (6) in case (2) under the monotonicity condition.

2. Exact solution for monotonic loading

As in the Hertz theory of elliptic contact between two bodies, the contact area $\omega(t)$ between the cartilage layers with the initial gap determined by Eq. (2) is elliptic with the semi-axes $a(t)$ and $b(t)$ changing with time. We have $a(t)\geq b(t)$, since $R_1\geq R_2$. The form of the ellipse $\Gamma(t)$ can be characterized by its aspect ratio $s=b(t)/a(t)$. Generally, $0<s\leq 1$ with the value $s=1$ corresponding to the circular contact area. We emphasize that the parameter s is constant during loading and depends only on the ratio R_2/R_1 through the relation

$$s=\sqrt{\sqrt{\left(\frac{R_1-R_2}{6R_1}\right)^2+\frac{R_2}{R_1}}-\frac{(R_1-R_2)}{6R_1}}. \qquad (9)$$

The evolution of the major semi-axes of the contact domain is governed by the equation

$$a(t)=\frac{1}{M_a(s)}\left(\frac{96}{\pi m}R\right)^{1/6}\left(F(t)+\chi\int_0^t F(\tau)\,d\tau\right)^{1/6}. \qquad (10)$$

Here, $R=2R_1R_2/(R_1+R_2)$ is a harmonic mean of the radii R_1 and R_2, while $M_a(s)$ is a dimensionless factor given by $M_a(s)=(8s^5/(3s^4+2s^2+3))^{1/6}$. It should be noted that in view of (3), the value of R does not depend on the orientation of the contacting surfaces determined by the angle α.

The relative approach between the subchondral bones is determined by the equations

$$\delta_0(t)=\frac{1}{2R}M_\delta(s)a^2(t), \quad M_\delta(s)=\frac{4s^2(s^2+1)}{3s^4+2s^2+3}. \qquad (11)$$

Now, if the contact load $F(t)$ is known, then Eqs. (10) and (11) allow to determine the quantities $a(t)$ and $\delta_0(t)$, respectively. In the case of the axisymmetric contact problem this coincides with the results obtained by Wu et al. (1997).

Finally, the contact pressure is calculated by means of the formula

$$P(x_1,x_2,t)=\left(1-\frac{x_1^2}{a^2(t)}-\frac{x_2^2}{s^2a^2(t)}\right)^2 p_0(t)$$

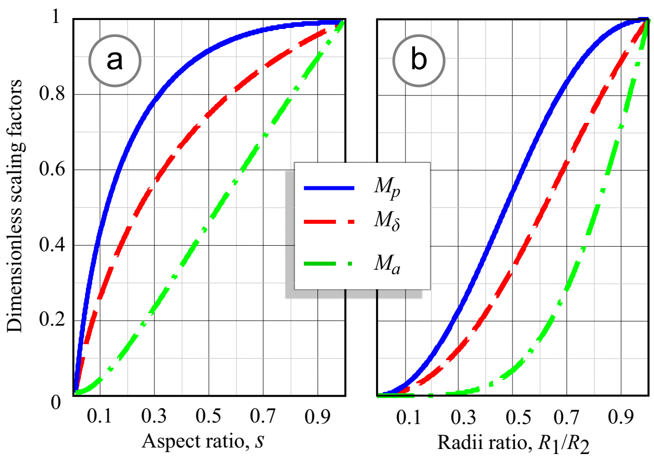

Fig. 2. Dimensionless scaling factors $M_a(s)$, $M_\delta(s)$, $M_p(s)$.

$$-\chi \int_{t_*(x_1,x_2)}^{t} \left(1-\frac{x_1^2}{a^2(\tau)}-\frac{x_2^2}{s^2a^2(\tau)}\right)^2 p_0(\tau)e^{-\chi(t-\tau)}\,d\tau, \tag{12}$$

where the auxiliary function $p_0(t)$ is given by

$$p_0(t)=\frac{m}{32R}M_p(s)a^4(t), \quad M_p(s)=\frac{8s^2}{3s^4+2s^2+3} \tag{13}$$

and $t_*(x_1,x_2)$ is the time when the contour of the contact zone first reaches the point (x_1,x_2). If, however, the point under consideration lies inside the initial contact domain ($(x_1,x_2)\in\omega(0)$) then $t_*(x_1,x_2)\equiv 0$. Non-zero quantity of $t_*(x_1,x_2)$ is determined by the equation $a^2(t_*)=x_1^2+x_2^2/s^2$, or in accordance with Eq. (10) by the following one:

$$F(t_*)+\chi\int_0^{t_*} F(\tau)\,d\tau = M_a^6(s)\frac{\pi m}{96R}\left(x_1^2+\frac{x_2^2}{s^2}\right)^3. \tag{14}$$

In the case of a stepwise loading, we have $F(t)=F_0$, and Eq. (14) admits the following closed-form solution for any $(x_1,x_2)\notin\omega(0)$:

$$t_*(x_1,x_2)=\frac{1}{\chi}\left(\frac{(x_1^2+s^{-2}x_2^2)^3}{a_0^6}-1\right), \tag{15}$$

where $a_0=\sqrt[6]{96RF_0/(\pi m)}/M_a(s)$ is the initial value of the major semi-axis. It should be noted that Eq. (5), $n=1,2$, holds true only when the cartilage thicknesses are small compared with the characteristic size of the contact area ($\max\{h_1,h_2\}\ll a_0$). That is why, the contact force F_0 (and, generally, $F(0)$) should not take too small values.

Thus, in the case of a stepwise loading, formula (12), where quantity $t_*(x_1,x_2)$ is determined by Eq. (15), represents the sought for solution to Eq. (6). Note that in the case of the axisymmetric problem the derived expression for the contact pressure coincides with the result obtained previously by Mishuris and Argatov (2010).

Fig. 2 shows the behavior of the quantities M_a, M_δ, and M_p with respect to the aspect ratio of the contact area, (a), and with respect to the radii ratio R_2/R_1, (b). It is interesting that this behavior for the different scaling factors is overall substantially similar.

3. Conclusion

The present study results in the exact closed-form solution to the three-dimensional contact problem for biphasic cartilage layers shaped as elliptic paraboloids which generalizes the axisymmetric solution found by Wu et al. (1997). The closed-form formulas (9)–(14) for evaluating the aspect ratio of the elliptic contact domain, its major semi-axis $a(t)$, the displacement parameter $\delta_0(t)$, the auxiliary parameter $p_0(t)$, and the contact pressure $P(x_1,x_2,t)$ in the spacial case (2) of contact of elliptic paraboloids constitute the main result of the present study. Detailed mathematical derivations of the solution can be found in the preprint (Argatov and Mishuris, 2010b).

Conflict of interest statement

The authors have no conflicts of interest.

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